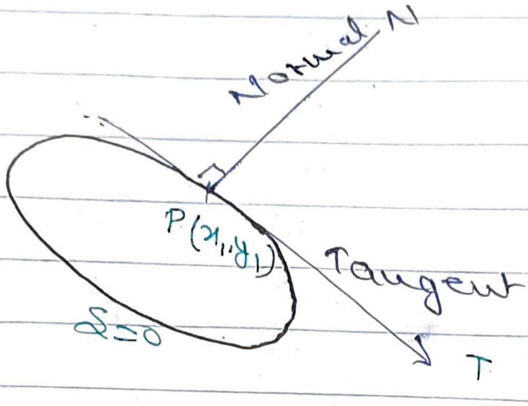
EQUATION OF NORMAL.

THEOREM:
To find the eq^n of normal to the conic

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
at any point $P(x_1, y_1)$.

Proof:



given eq^n of conic is
$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ —— (1)

diff (1) partially w.r.t x & y we get
$$\frac{\partial S}{\partial x} = 2(ax + hy + g)$$

$$\frac{\partial S}{\partial y} = 2(hx + by + f)$$

Let PN be the normal to the conic (1)
at any point $P(x_1, y_1)$
then its eq^n is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

$$= +\left(\frac{\partial S/\partial x}{\partial S/\partial y}\right)_{(x_1,y_1)} (x-x_1)$$

$$= +\left\{\frac{\partial S/\partial y}{\partial S/\partial x}\right\}_{(x_1,y_1)} (x-x_1)$$

$$y-y_1 = +\frac{2(hx_1+by_1+f)}{2(ax_1+hy_1+g)}(x-x_1)$$

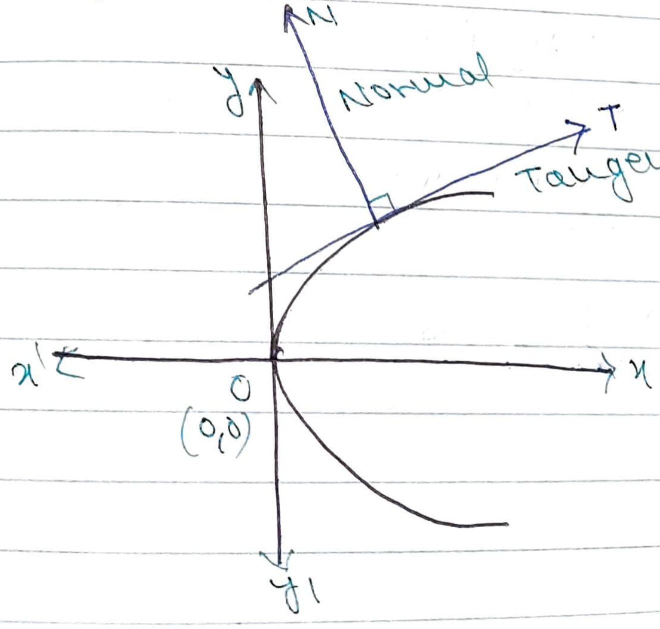
$$\boxed{\frac{y-y_1}{(hx_1+by_1+f)} = \frac{x-x_1}{ax_1+hy_1+g}}$$

which is the required eqn of Normal to the conic ① at pt. (x_1,y_1).

PARTICULAR CASES:

A) To find the eqn of Normal to the parabola $y^2=4ax$ at any point (x_1,y_1)

$$\frac{y-y_1}{-2y_1} = \frac{x-x_1}{+4a}$$



given eqn of parabola is

$S \equiv y^2 = 4ax$

$S \equiv y^2 - 4ax = 0$ ——— (1)

diff (1) partially w.r.t. x & y

$\frac{\partial S}{\partial x} = -4a$

$\frac{\partial S}{\partial y} = 2y$

let PN be normal to the parabola (1) at any point $P(x_1, y_1)$ then its eqn is

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

$$= \frac{-1}{-\left(\frac{\partial S/\partial x}{\partial S/\partial y}\right)_{(x_1, y_1)}} (x - x_1)$$

$$= \left(\frac{\partial S/\partial y}{\partial S/\partial x}\right)_{x_1, y_1} (x - y_1$$

$$y - y_1 = \frac{2y_1}{-4a} (x - x_1)$$

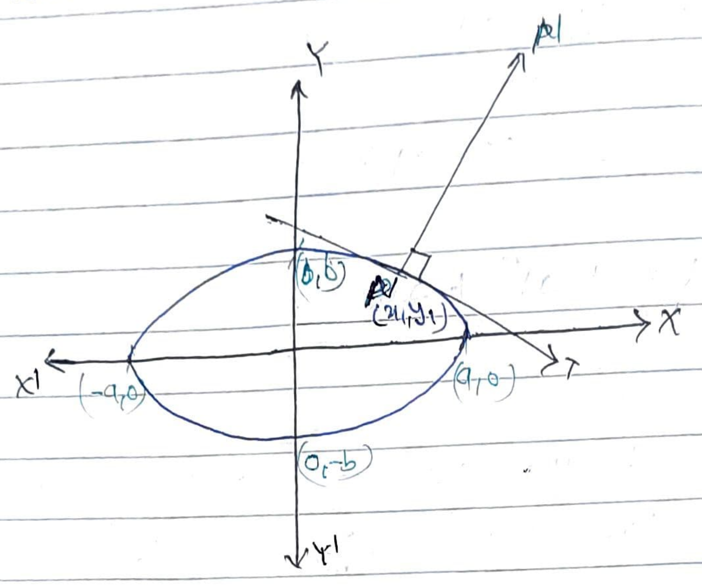
$$y - y_1 = -\frac{y_1}{2a} (x - x_1)$$

$$\boxed{\frac{x - x_1}{2a} + \frac{y - y_1}{y_1} = 0.}$$

B) To find the equation of normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{at } (x_1, y_1)$$

Solution:



given equation of ellipse is

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

diff. (1) partially w.r.t x and y we get

$$\frac{\partial s}{\partial x} = \frac{2x}{a^2}$$

$$\frac{\partial s}{\partial y} = \frac{2y}{b^2}$$

Let PN be normal to the parabola (1) at any point $P(x_1, y_1)$ its eq^n eq^n

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

$$y - y_1 = \frac{-1}{\left(-\frac{\frac{\partial S}{\partial x}}{\frac{\partial S}{\partial y}}\right)_{(x_1, y_1)}} (x - x_1)$$

$$= \left(\frac{\partial S/\partial y}{\partial S/\partial x}\right)_{(x_1, y_1)} (x - x_1)$$

$$y - y_1 = \left(\frac{2y_1/b^2}{2x_1/a^2}\right) (x - x_1)$$

$$\text{or, } y - y_1 = \frac{y_1}{x_1}\frac{a^2}{b^2} (x - x_1)$$

$$\text{or, } \boxed{\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}}$$

$\therefore$ which is the required equation of Normal

Corollary:

The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\phi, b\sin\phi)$ is

$$\boxed{ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2}$$